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A POSTERIORI ESTIMATES FOR EULER AND NAVIER-STOKES EQUATIONS

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Abstract. The first two sections of this work review the framework of [6] for approximate solutions of the incompressible Euler or Navier-Stokes (NS) equations on a torus $\mathbf{T}^d$, in a Sobolev setting. This approach starts from an approximate solution u_{a} of the Euler/NS Cauchy problem and, analyzing it *a posteriori*, produces estimates on the interval of existence of the exact solution u and on the distance between u and u_{a}. The next two sections present an application to the Euler Cauchy problem, where u_{a} is a Taylor polynomial in the time variable t; a special attention is devoted to the case $d = 3$, with an initial datum for which Behr, Nečas and Wu have conjectured a finite time blowup [1]. These sections combine the general approach of [6] with the computer algebra methods developed in [9]; choosing the Behr-Nečas-Wu datum, and using for u_{a} a Taylor polynomial of order 52, a rigorous lower bound is derived on the interval of existence of the exact solution u, and an estimate is obtained for the H^3 Sobolev distance between $u(t)$ and $u_{\mathsf{a}}(t)$.

1. Preliminaries.

Throughout this work we fix a space dimension $d \in \{2, 3, ...\}$; in the application of section 4 we will put $d = 3$. For a, b in $\mathbf{R}^d$ or $\mathbf{C}^d$ we put $a \bullet b := \sum_{r=1}^{d} a_r b_r$ and $|a| := \sqrt{\overline{a} \bullet a}$, with $^{-}$ indicating the complex conjugate.

Let us consider the d-dimensional torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$; we denote with $(e_k)_{k \in \mathbf{Z}^d}$ the Fourier basis made of the functions $e_k : \mathbf{T}^d \to \mathbf{C}$, $e_k(x) := (2\pi)^{-d/2} e^{ik \bullet x}$. Here and in the sequel, "a vector field on $\mathbf{T}^d$" means "an $\mathbf{R}^d$-valued distribution on $\mathbf{T}^d$" (see, e.g., [5]); we write $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$ for the space of such distributions. Any $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$, with coefficients $v_k \in \mathbf{C}^d$ such that $\overline{v_k} = v_{-k}$.

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In the sequel $\mathbb{L}^p(\mathbf{T}^d) \equiv \mathbb{L}^p$ denotes the space of L^p vector fields $\mathbf{T}^d \to \mathbf{R}^d$. For all $n \in \mathbf{R}$ we introduce the Sobolev space of zero mean, divergence free vector fields of order n; this is

$$\begin{aligned}(1.1)\qquad \mathbb{H}^n_{\Sigma 0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{\Sigma 0} &:= \Big\{ v \in \mathbb{D}' \mid \int_{\mathbf{T}^d} v\, dx = 0,\ \operatorname{div} v = 0,\ \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2 \ \Big\} \\ &= \Big\{ v \in \mathbb{D}' \mid v_0 = 0,\ k\bullet v_k = 0 \text{ for all } k,\ \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k|^2 < +\infty \ \Big\}\end{aligned}$$

(in the above, $\int_{\mathbf{T}^d} v\, dx$ indicates the action of v on the test function 1 and $\sqrt{-\Delta}^{\,n} v := \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^n v_k e_k$). $\mathbb{H}^n_{\Sigma 0}$ is a Hilbert space with the inner product and the norm

$$(1.2)\quad \langle v | w \rangle_n := \langle \sqrt{-\Delta}^{\,n} v | \sqrt{-\Delta}^{\,n} w \rangle_{L^2} = \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} \overline{v_k} \bullet w_k, \qquad \|v\|_n := \sqrt{\langle v | v \rangle_n}\ ;$$

if $m \leqslant n$ then $\mathbb{H}^n_{\Sigma 0} \subset \mathbb{H}^m_{\Sigma 0}$.

1.1. **The bilinear map for the Euler or Navier-Stokes (NS) equations.** Consider two vector fields v, w on $\mathbf{T}^d$ such that $v \in \mathbb{L}^2$ and $\partial_r w \in \mathbb{L}^2$ for $r = 1, ..., d$; then we have a well defined vector field $v \bullet \partial w \in \mathbb{L}^1$ of components $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$; we can apply to this the Leray projection $\mathfrak{L}$, sending $\mathbb{D}'$ onto the space of divergence free vectors fields, and form the vector field

$$(1.3)\qquad \mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w)\ .$$

The bilinear map $\mathcal{P}$: $(v, w) \mapsto \mathcal{P}(v, w)$, which is a main character of the incompressible Euler/NS equations, is known to possess the following properties:
(i) For each $n > d/2$, $\mathcal{P}$ is continuous from $\mathbb{H}^n_{\Sigma 0} \times \mathbb{H}^{n+1}_{\Sigma 0}$ to $\mathbb{H}^n_{\Sigma 0}$; so, there is a constant $K_{nd} \equiv K_n$ such that

$$(1.4)\qquad \|\mathcal{P}(v, w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1} \quad \text{for } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\ .$$

(ii) For each $n > d/2 + 1$, there is a constant $G_{nd} \equiv G_n$ such that

$$(1.5)\qquad |\langle \mathcal{P}(v, w) | w \rangle_n| \leqslant G_n \|v\|_n \|w\|_n^2 \quad \text{for } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\ .$$

The result (ii) is due to Kato, see [3]. In papers [7] [8], (1.4) and (1.5) are called the "basic inequality" and the "Kato inequality", respectively; in these papers, computable upper and lower bounds are given for the sharp constants appearing therein. From here to the end of this work, K_n and G_n are constants fulfilling the previous inequalities (and not necessarily sharp). From [7] [8] we know that we can take

$$(1.6)\qquad K_3 = 0.323\ , \qquad G_3 = 0.438 \qquad \text{if } d = 3\ ;$$

these values will be useful in the sequel.

1.2. **The Euler/NS Cauchy problem.** Let us fix a Sobolev order

$$(1.7)\qquad n \in \Big(\frac{d}{2} + 1, +\infty\Big)\ .$$

We choose a "viscosity coefficient" $\nu \in [0, +\infty)$, and put

$$(1.8)\qquad \overline{\nu} := \begin{cases} 1 & \text{if } \nu = 0, \\ 2 & \text{if } \nu > 0. \end{cases}$$

Furthermore, we choose a "forcing"

$$(1.9)\qquad f \in C([0, +\infty), \mathbb{H}^n_{\Sigma 0})$$

and an initial datum

$$u_0 \in \mathbb{H}^{n+\overline{\nu}}_{\Sigma 0} \ . \tag{1.10}$$

Definition 1.1. *The Cauchy problem for the (incompressible) fluid with viscosity ν, initial datum u_0 and forcing f is the following:*

$$\text{Find } u \in C([0,T), \mathbb{H}^{n+\overline{\nu}}_{\Sigma 0}) \cap C^1([0,T), \mathbb{H}^{n}_{\Sigma 0}) \quad \text{such that} \tag{1.11}$$
$$\frac{du}{dt} = \nu \Delta u + \mathcal{P}(u,u) + f \ , \qquad u(0) = u_0$$

(with $T \in (0, +\infty]$, depending on u). As usually, we speak of the "Euler Cauchy problem" if $\nu = 0$, and of the "NS Cauchy problem" if $\nu > 0$.

It is known [4] that the above Cauchy problem has a unique maximal (i.e., non extendable) solution; any solution is a restriction of the maximal one.

2. Approximate solutions of the Euler/NS Cauchy problem

We consider again the Cauchy problem (1.11), for given n, ν, f, u_0 as in the previous section. The definitions and the theorem that follow are taken from [6].

Definition 2.1. *An approximate solution of problem (1.11) is any map $u_{\mathtt{a}} \in C([0,T_{\mathtt{a}}), \mathbb{H}^{n+\overline{\nu}}_{\Sigma 0}) \cap C^1([0,T_{\mathtt{a}}), \mathbb{H}^{n}_{\Sigma 0})$ (with $T_{\mathtt{a}} \in (0,+\infty]$). Given such a function, we stipulate (i) (ii).*
(i) The differential error of $u_{\mathtt{a}}$ is

$$\frac{du_{\mathtt{a}}}{dt} - \nu \Delta u_{\mathtt{a}} - \mathcal{P}(u_{\mathtt{a}}, u_{\mathtt{a}}) - f \ \in C([0,T_{\mathtt{a}}), \mathbb{H}^{n}_{\Sigma 0}) \ ; \tag{2.1}$$

the datum error is

$$u_{\mathtt{a}}(0) - u_0 \in \mathbb{H}^{n+\overline{\nu}}_{\Sigma 0} \ . \tag{2.2}$$

(ii) Let $m \in \mathbf{R}, m \leqslant n$. A differential error estimator of order m for $u_{\mathtt{a}}$ is a function

$$\epsilon_m \in C([0,T_{\mathtt{a}}), [0,+\infty)) \quad \text{such that} \tag{2.3}$$
$$\| (\frac{du_{\mathtt{a}}}{dt} - \nu \Delta u_{\mathtt{a}} - \mathcal{P}(u_{\mathtt{a}}, u_{\mathtt{a}}) - f)(t) \|_m \leqslant \epsilon_m(t) \quad \text{for } t \in [0, T_{\mathtt{a}}) \ .$$

Let $m \in \mathbf{R}$, $m \leqslant n + \overline{\nu}$. A datum error estimator of order m for $u_{\mathtt{a}}$ is a real number

$$\delta_m \in [0,+\infty) \quad \text{such that} \quad \|u_{\mathtt{a}}(0) - u_0\|_m \leqslant \delta_m \ ; \tag{2.4}$$

a growth estimator of order m for $u_{\mathtt{a}}$ is a function

$$\mathcal{D}_m \in C([0,T_{\mathtt{a}}), [0,+\infty)) \quad \text{such that} \quad \|u_{\mathtt{a}}(t)\|_m \leqslant \mathcal{D}_m(t) \quad \text{for } t \in [0,T_{\mathtt{a}}) \ . \tag{2.5}$$

In particular $\epsilon_m(t) := \|(du_{\mathtt{a}}/dt - \nu \Delta u_{\mathtt{a}} - \mathcal{P}(u_{\mathtt{a}}, u_{\mathtt{a}}) - f)(t)\|_m$, $\delta_m := \|u_{\mathtt{a}}(0) - u_0\|_m$ and $\mathcal{D}_m(t) := \|u_{\mathtt{a}}(t)\|_m$ will be called the tautological estimators of order m for the differential error, the datum error and the growth of $u_{\mathtt{a}}$.

From here to the end of the section we consider an approximate solution $u_{\mathtt{a}}$ of problem (1.11) of domain $[0, T_{\mathtt{a}})$; this is assumed to possess differential, datum error and growth estimators of orders n or $n+1$, indicated with $\epsilon_n, \delta_n, \mathcal{D}_n, \mathcal{D}_{n+1}$.

Definition 2.2. *Let $\mathcal{R}_n \in C([0,T_{\mathtt{c}}), [0,+\infty))$, with $T_{\mathtt{c}} \in (0, T_{\mathtt{a}}]$. This function is said to fulfil the control inequalities if*

$$\frac{d^+ \mathcal{R}_n}{dt} \geqslant -\nu \mathcal{R}_n + (G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \mathcal{R}_n + G_n \mathcal{R}_n^2 + \epsilon_n \text{ everywhere on } [0, T_{\mathtt{c}}), \tag{2.6}$$

$$\mathcal{R}_n(0) \geqslant \delta_n \ . \tag{2.7}$$

In the above d^+/dt indicates the right, upper Dini derivative: so, for all $t \in [0, T_{\mathsf{c}})$, $(d^+\mathcal{R}_n/dt)(t) := \limsup_{h\to 0^+}[\mathcal{R}_n(t+h) - \mathcal{R}_n(t)]/h$.

Proposition 2.1. *Assume there is a function $\mathcal{R}_n \in C([0, T_{\mathsf{c}}), [0, +\infty))$ fulfilling the control inequalities; consider the maximal solution u of the Euler/NS Cauchy problem (1.11), and denote its domain with $[0, T)$. Then*

$$T \geqslant T_{\mathsf{c}} \ , \tag{2.8}$$

$$\|u(t) - u_{\mathsf{a}}(t)\|_n \leqslant \mathcal{R}_n(t) \qquad \textit{for } t \in [0, T_{\mathsf{c}}) \ . \tag{2.9}$$

Proof. (Sketch) One introduces the function $\|u - u_{\mathsf{a}}\|_n : t \in [0, T) \cap [0, T_{\mathsf{a}}) \mapsto \|u(t) - u_{\mathsf{a}}(t)\|_n$ and shows that $d^+\|u - u_{\mathsf{a}}\|_n/dt \leqslant -\nu \, \|u - u_{\mathsf{a}}\|_n + (G_n\mathcal{D}_n + K_n\mathcal{D}_{n+1}) \, \|u - u_{\mathsf{a}}\|_n + G_n\|u - u_{\mathsf{a}}\|_n^2 + \epsilon_n$, (see Lemma 4.2 of [6], greatly indebted to [2]); moreover, $\|u(0) - u_{\mathsf{a}}(0)\|_n \leqslant \delta_n$. From here, from the control inequalities (2.6) (2.7) and from the Čaplygin comparison lemma one infers that $\|u(t) - u_{\mathsf{a}}(t)\|_n \leqslant \mathcal{R}_n(t)$ for $t \in [0, T) \cap [0, T_{\mathsf{c}})$. Finally, it is $T \geqslant T_{\mathsf{c}}$; in fact, it it were $T < T_{\mathsf{c}}$, the previous inequality about u, u_{a} and $\mathcal{R}_n$ would imply $\limsup_{t\to T^-} \|u(t)\|_n < +\infty$, a fact contradicting the maximality assumption for u. See [6] for more details. □

Paper [6] presents some applications of the previous proposition, dealing with both the Euler case $\nu = 0$ and the NS case $\nu > 0$; a special attention is devoted therein to the approximate solutions u_{a} provided by the Galerkin method.

In this work we present an application of Proposition 2.1 to the Euler case $\nu = 0$, choosing for u_{a} a polynomial in the time variable t. In the next section we present this procedure in general, giving the error estimators for approximate solutions of this kind; in the last section we apply the procedure choosing for u_0 the so-called Behr-Nečas-Wu initial datum.

3. Polynomial approximate solutions for the Euler equations

Let us recall that $n \in (d/2 + 1, +\infty)$, and consider the Euler Cauchy problem with a datum $u_0 \in \mathbb{H}^{n+1}_{\Sigma 0}$ and zero external forcing:

$$\text{Find } u \in C([0, T), \mathbb{H}^{n+1}_{\Sigma 0}) \cap C^1([0, T), \mathbb{H}^{n}_{\Sigma 0}) \quad \text{such that} \qquad \frac{du}{dt} = \mathcal{P}(u, u) \ , \qquad u(0) = u_0 \ . \tag{3.1}$$

Let us choose an order $N \in \{0, 1, 2, ...\}$ and consider as an approximate solution for (3.1) a polynomial of degree N in time, of the form

$$u^N : [0, +\infty) \to \mathbb{H}^{n+1}_{\Sigma 0} \ , \qquad t \mapsto u^N(t) := \sum_{j=0}^{N} u_j t^j \qquad (u_j \in \mathbb{H}^{n+1}_{\Sigma 0} \text{ for all } j) \ . \tag{3.2}$$

Here u_0 is the initial datum, and u_j is to be determined for $j = 1, ..., N$.

Proposition 3.1. *(i) Let u^N be as in (3.2). The datum and differential errors of u^N are*

$$u^N(0) - u_0 = 0 \ ; \tag{3.3}$$

$$\frac{du^N}{dt}(t) - \mathcal{P}(u^N, u^N)(t) \tag{3.4}$$

$$= \sum_{j=0}^{N-1} \Big[(j+1)u_{j+1} - \sum_{\ell=0}^{j} \mathcal{P}(u_\ell, u_{j-\ell})\Big] t^j - \sum_{j=N}^{2N} \Big[\sum_{\ell=j-N}^{N} \mathcal{P}(u_\ell, u_{j-\ell})\Big] t^j \ .$$

(ii) In particular, assume

$$u_{j+1} = \frac{1}{j+1} \sum_{\ell=0}^{j} \mathcal{P}(u_\ell, u_{j-\ell}) \quad \text{for } j = 0, ..., N-1 \ ; \tag{3.5}$$

then

$$\frac{du^N}{dt}(t) - \mathcal{P}(u^N, u^N)(t) = - \sum_{j=N}^{2N} \Big[\sum_{\ell=j-N}^{N} \mathcal{P}(u_\ell, u_{j-\ell}) \Big] t^j = O(t^N) \text{ for } t \to 0 \ . \tag{3.6}$$

(iii) If (3.5) is used to define recursively $u_1, ..., u_N$, it produces a sequence of elements of $\mathbb{H}^{n+1}_{\Sigma 0}$ under the condition $u_0 \in \mathbb{H}^{n+1+N}_{\Sigma 0}$. More precisely, from $u_0 \in \mathbb{H}^{n+1+N}_{\Sigma 0}$ it follows $u_j \in \mathbb{H}^{n+1+N-j}_{\Sigma 0} \subset \mathbb{H}^{n+1}_{\Sigma 0}$ for $j = 1, ..., N$.
(iv) Let $u_0 \in \mathbb{H}^{n+N+1}_{\Sigma 0}$ and use (3.5) to define u_j for $j = 1, ..., N$. Then

$$\| \frac{du^N}{dt}(t) - \mathcal{P}(u^N, u^N)(t) \|_n \leqslant \epsilon_n(t) \qquad \text{for } t \in [0, +\infty) \ , \tag{3.7}$$

$$\epsilon_n(t) := K_n \sum_{j=N}^{2N} \Big[\sum_{\ell=j-N}^{N} \|u_\ell\|_n \|u_{j-\ell}\|_{n+1} \Big] t^j \text{ for } t \in [0, +\infty) \ . \tag{3.8}$$

Proof. (i) (3.3) is obvious; let us prove (3.4). To this purpose, we note that

$$\frac{du^N}{dt} - \mathcal{P}(u^N, u^N) = \frac{d}{dt}\Big(\sum_{\ell=0}^{N} u_\ell t^\ell \Big) - \mathcal{P}\Big(\sum_{\ell=0}^{N} u_\ell t^\ell, \sum_{h=0}^{N} u_h t^h \Big)$$

$$= \sum_{\ell=1}^{N} \ell u_\ell t^{\ell-1} - \sum_{\ell,h=0}^{N} \mathcal{P}(u_\ell, u_h) t^{\ell+h} = \sum_{j=0}^{N-1} (j+1) u_{j+1} t^j - \sum_{j=0}^{2N} \Big[\sum_{(\ell,h) \in I_{Nj}} \mathcal{P}(u_\ell, u_h) \Big] t^j \ ,$$

$$I_{Nj} := \{ (\ell, h) \in \{0, ..., N\}^2 \mid \ell + h = j \} \ .$$

One easily checks that

$$j \in \{0, ..., N-1\} \ \Rightarrow \ I_{Nj} = \{ (\ell, j-\ell) \mid \ell \in \{0, ..., j\} \} \ ,$$

$$j \in \{N, ..., 2N\} \ \Rightarrow \ I_{Nj} = \{ (\ell, j-\ell) \mid \ell \in \{j-N, ..., N\} \} \ ;$$

this readily yields the thesis (3.4).
(ii) Obvious.
(iii) Let $u_0 \in \mathbb{H}^{n+1+N}_{\Sigma 0}$ and define $u_1, ..., u_N$ via the recursion relation (3.5). Then $u_1 = \mathcal{P}(u_0, u_0) \in \mathbb{H}^{n+N}_{\Sigma 0}$, $u_2 = (1/2)\mathcal{P}(u_0, u_1) + (1/2)\mathcal{P}(u_1, u_0) \in \mathbb{H}^{n+N-1}_{\Sigma 0}$, etc. .
(iv) Eq. (3.6) implies $\|(du^N/dt)(t) - \mathcal{P}(u^N, u^N)(t)\|_n \leqslant \sum_{j=N}^{2N} \Big[\sum_{\ell=j-N}^{N} \|\mathcal{P}(u_\ell, u_{j-\ell})\|_n \Big] t^j$. On the other hand Eq. (1.4) gives $\|\mathcal{P}(u_\ell, u_{j-\ell})\|_n \leqslant K_n \|u_\ell\|_n \|u_{j-\ell}\|_{n+1}$, whence the thesis (3.7) (3.8). $\square$

4. A special case of the previous framework: the Euler equations on $\mathbf{T}^3$, with the Behr-Nečas-Wu initial datum.

In this section we consider the Euler Cauchy problem (3.1) with space dimension and Sobolev order

$$d = 3 \;, \qquad n = 3 \;; \tag{4.1}$$

the initial datum is

$$u_0 := \sum_{k=\pm a, \pm b, \pm c} u_{0k} e_k \;, \tag{4.2}$$

$$a := (1,1,0), \; b := (1,0,1), \; c := (0,1,1) \;;$$

$$u_{0,\pm a} := (2\pi)^{3/2}(1,-1,0), \quad u_{0,\pm b} := (2\pi)^{3/2}(1,0,-1), \quad u_{0,\pm c} := (2\pi)^{3/2}(0,1,-1)$$

(of course, being a Fourier polynomial, u_0 belongs to $\mathbb{H}^m_{\Sigma 0}$ for each $m \in \mathbf{R}$). The above initial datum is considered by Behr, Nečas and Wu in [1]; it is analyzed with a similar attitude in [9] (and, from a different viewpoint, in [6]). In both papers [1] [9], attention is fixed on the function $u^N(t) = \sum_{j=0}^N u_j t^j$ for a rather large value of N, where the u_j's are determined for $j = 1, ..., N$ by the recursion relation (3.5). The u_j's are Fourier polynomials and can be calculated exactly by computer algebra methods; such computations are performed in [1] for $N = 35$, and in [9] up to $N = 52$ (using, respectively, the C^{++} and the Python languages).

The Python program of [9] gives exact expressions for the u_j's, whose Fourier components are rational (up to factors $(2\pi)^{3/2}$); for large j, these expressions are terribly complicated. Here, to give a partial illustration of such Python computations we consider the Fourier components $u^{52}_k(t)$ for $k = (1,1,0)$ and $k = (0,0,2)$, and report the graphs of the functions $t \mapsto |u^{52}_k(t)|$ for these wave vectors: see Figures 1 and 2.

In both papers [1] [9], computations are used to get hints about $\lim_{N \to +\infty} u^N$, giving the exact solution of the Euler Cauchy problem on the time interval where the limit exists; however the statements of [1] [9] rely on the assumption that certain facts on the $N \to +\infty$ limit can be extrapolated from u^{35} or u^{52}. In particular [1] makes the conjecture, disputed in [9], that the solution of the Euler Cauchy problem blows up for $t \to \tau^-$, with $\tau \simeq 0.32$

In the present work we make no conjecture or extrapolation about the $N \to +\infty$ limit and just consider the function u^{52} of [9] according to the general framework of approximate solutions and control inequalities. This approach produces:
(i) a rigorous lower bound on the interval of existence of the exact solution u of the $(d = 3, n = 3)$ Cauchy problem (3.1);
(ii) a bound on $\|u(t) - u^{52}(t)\|_3$.

To get these results we regard u^{52} as an approximate solution of (3.1), using the tautological datum error and growth estimators

$$\delta_3 := 0 \;; \quad \mathcal{D}_3(t) := \|u^{52}(t)\|_3 \;, \; \mathcal{D}_4(t) := \|u^{52}(t)\|_4 \text{ for } t \in [0, +\infty) \tag{4.3}$$

(concerning δ_3, we recall that $u^{52}(0) - u_0 = 0$). For $m = 3, 4$ one has $\mathcal{D}_m(t) = (2\pi)^{3/2}[\sum_{j=0}^{52} d_{mj} t^{2j}]^{1/2}$ where the d_{mj} are rational coefficients; the Python program employed for our work [9] computes exactly these coefficients. For $m = 3$ these coefficients are reported in [9], in a 16-digits decimal representation (see Eq. (5.12) of [9], not containing the factor $(2\pi)^{3/2}$ due to a different normalization of the norm

$\| \ \|_3$); we have no room to report here the coefficients of the $m = 4$ case. Figures 3 and 4 contain the graphs of the functions $t \mapsto \mathcal{D}_3(t), \mathcal{D}_4(t)$.

Let us pass to the differential error estimator for u^{52}; we use for it the function ϵ_3 defined by (3.8) with $n = 3$ and $K_3 = 0.323$, see (1.6). ϵ_3 is computed exactly by our Python program; again, the explicit expression is too complicated to be reported. (The tautological error estimator $\epsilon_3^*(t) := \|(du^{52}/dt)(t) - \mathcal{P}(u^{52}, u^{52})(t)\|_3$ $= \| \sum_{j=52}^{104} \Big[\sum_{\ell=j-52}^{52} \mathcal{P}(u_\ell, u_{j-\ell}) \Big] t^j \|_3$ is more accurate, but it has an even more complicated expression; its calculation by computer algebra is too expensive.)

For the graph of ϵ_3 and some information on its numerical values, see Figure 5 and its caption. With the previous ingredients, we build the following "control Cauchy problem": find $\mathcal{R}_3$ such that

$$\mathcal{R}_3 \in C^1([0, T_{\mathsf{c}}), \mathbf{R}), \ \frac{d\mathcal{R}_3}{dt} = (G_3\mathcal{D}_3 + K_3\mathcal{D}_4)\mathcal{R}_3 + G_3\mathcal{R}_3^2 + \epsilon_3, \ \mathcal{R}_3(0) = 0 \tag{4.4}$$

($G_3 = 0.438$, see again (1.6)). This control problem has a unique maximal solution $\mathcal{R}_3$, which is strictly increasing and thus positive for $t \in (0, T_{\mathsf{c}})$. Of course, this $\mathcal{R}_3$ fulfils as equalities Eqs. (2.6) (2.7) (with $\nu = 0$).

Once we have $\mathcal{R}_3 : [0, T_{\mathsf{c}}) \to [0, +\infty)$, due to Proposition 2.1 we can grant that:
(i) The maximal solution u of the ($n = 3$) Euler Cauchy problem (3.1) is defined on an interval $[0, T)$ with $T \geqslant T_{\mathsf{c}}$;
(ii) It is

$$\|u(t) - u^{52}(t)\|_3 \leqslant \mathcal{R}_3(t) \qquad \text{for } t \in [0, T_{\mathsf{c}}) \ . \tag{4.5}$$

The function $\mathcal{R}_3$ can be determined numerically by a cheap computation using any package for ODEs, e.g. Mathematica (the result is reliable, since (4.4) is the Cauchy problem for a simple ODE in one dimension). This numerical computation indicates that the (maximal) domain of $\mathcal{R}_3$ is $[0, T_{\mathsf{c}})$, with

$$T_{\mathsf{c}} = 0.242...; \tag{4.6}$$

After having been extremely small for most of the time between 0 and T_{c}, $\mathcal{R}_3(t)$ diverges abruptly for $t \to T_{\mathsf{c}}^-$; for the graph of this function and some information on its numerical values, see Figure 6 and its caption. Due to (4.6), we can grant that the solution u of the Euler Cauchy problem (1.11) exists on a time interval of length $T \geqslant 0.242$ (this is four times larger than the lower bound on T obtained in [6] using a Galerkin approximate solution).

Eq. (4.5) and the previously described behavior of $\mathcal{R}_3$ ensure that $u^{52}(t)$ approximates with extreme precision $u(t)$ on most of the time interval $[0, T_{\mathsf{c}})$. We remark that (4.5) can be used to infer other interesting estimates about $u - u^{52}$, e.g.,

$$|u_k(t) - u_k^{52}(t)| \leqslant \frac{\mathcal{R}_3(t)}{|k|^3} \qquad \text{for } k \in \mathbf{Z}^3 \setminus \{0\}, \, t \in [0, T_{\mathsf{c}}) \ ; \tag{4.7}$$

this follows from (4.5) and from the elementary inequality $|v_k| \leqslant \|v\|_3/|k|^3$, holding for all $v \in \mathbb{H}^3_{\Sigma 0}$ and $k \in \mathbf{Z}^3 \setminus \{0\}$ (recall that $\|v\|_3^2 = \sum_{k \in \mathbf{Z}^3 \setminus \{0\}} |k|^6 |v_k|^2$).

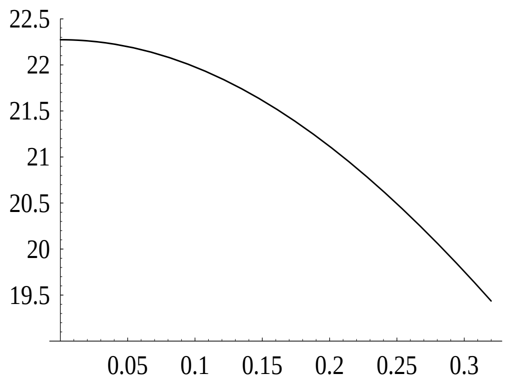

Figure 1. Plot of $|u^{52}_{(1,1,0)}(t)|$ for $t \in [0, 0.32]$.

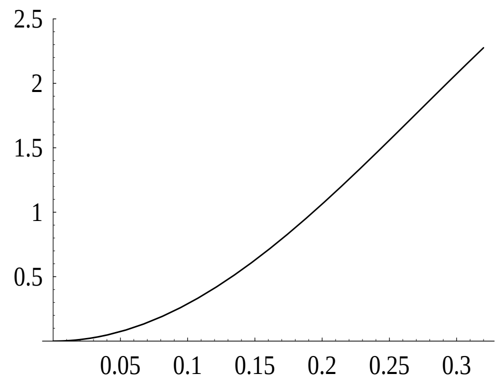

Figure 2. Plot of $|u^{52}_{(0,0,2)}(t)|$ for $t \in [0, 0.32]$.

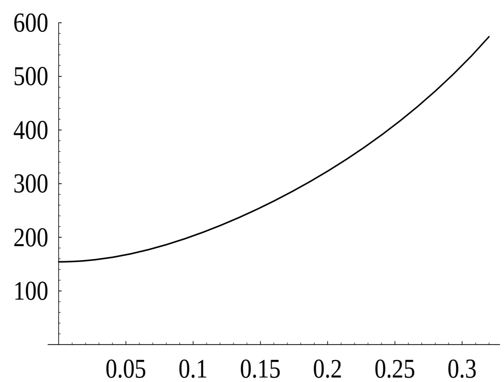

Figure 3. Plot of $\mathcal{D}_3(t)$ for $t \in [0, 0.32]$.

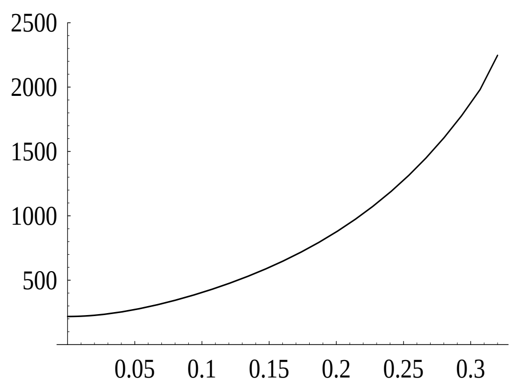

Figure 4. Plot of $\mathcal{D}_4(t)$ for $t \in [0, 0.32]$.

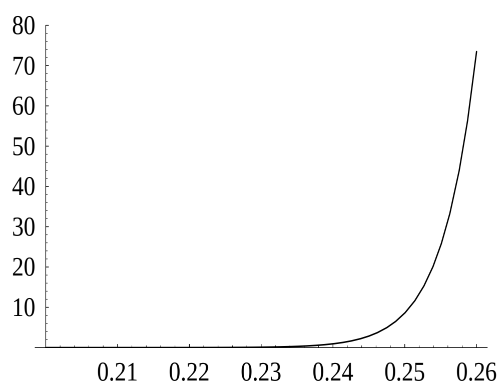

Figure 5. Plot of $\epsilon_3(t)$ for $t \in [0.20, 0.26]$. One has: $\epsilon_3(t) < 10^{-20}$ for $t \in [0, 0.10]$; $\epsilon_3(t) < 10^{-4}$ for $t \in (0.10, 0.20]$; $\epsilon_3(t) < 10^{-3}$ for $t \in (0.20, 0.21]$; $\epsilon_3(t) < 8.6 \times 10^{-3}$ for $t \in (0.21, 0.22]$; $\epsilon(t) < 0.094$ for $t \in (0.22, 0.23]$; $\epsilon(t) < 0.93$ for $t \in (0.23, 0.24]$, $\epsilon_3(t) < 8.6$ for $t \in (0.24, 0.25]$; $\epsilon_3(t) < 74$ for $t \in (0.25, 0.26]$.

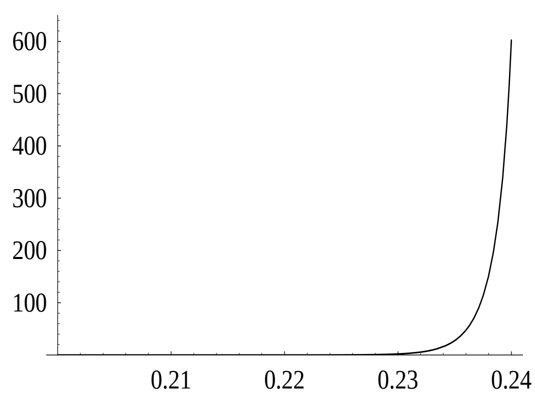

Figure 6. Plot of $\mathcal{R}_3(t)$ for $t \in [0.20, 0.24]$. One has: $\mathcal{R}_3(t) < 2 \times 10^{-6}$ for $t \in [0, 0.20]$; $\mathcal{R}_3(t) < 1.2 \times 10^{-4}$ for $t \in (0.20, 0.21]$; $\mathcal{R}_3(t) < 0.013$ for $t \in (0.21, 0.22]$; $\mathcal{R}_3(t) < 2$ for $t \in (0.22, 0.23]$; $\mathcal{R}_3(t) < 610$ for $t \in (0.23, 0.24]$.

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